Algebraic Number Theory
IInd Semesteral examination 2011
M.Math. II
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Q 1.
Discuss the steps used to deduce that the ray class group for a modulus of a number field, is finite, of order a multiple of the class number.

OR

Consider a number field K with r_1 real, and $2r_2$ nonreal embeddings into $\mathbf{C}$ over $\mathbf{Q}$. Write down a map which shows how $\mathcal{O}_K$ can be regarded as a lattice in $\mathbf{R}^n$ where $n = r_1 + 2r_2$. Further, prove that the volume of a fundamental parallelotope for this lattice is $\frac{\sqrt{|d_K|}}{2^{r_2}}$.

Q 2.
Let L/K be a cyclic (Galois) extension of number fields. Show that the set of primes of K which are inert (that is, remain prime) in L has a Dirichlet density and compute it.

OR

Let L/K be an abelian extension of number fields. Show that the set of primes of K which split completely in L has a Dirichlet density and compute it.

Q 3.
Prove that a p-adic number $\sum_{i \geq -n} c_i p^i$ is rational if, and only if, the sequence $\{c_i\}$ is eventually periodic. Determine the p-adic expansion of $2/7$.

OR

Obtain the order of the group of roots of unity in $\mathbf{Q}_p$ for some odd prime p. Deduce that $\mathbf{Q}_p$ is not isomorphic to $\mathbf{Q}_q$ for odd primes $p \neq q$.

Q 4.
For $K = \mathbf{Q}(2^{1/3})$, assume $\mathcal{O}_K = \mathbf{Z}[2^{1/3}]$. Compute the ramification indices e_i and residue field degrees f_i of the prime/primes lying over 5.

OR

Let $K = \mathbf{Q}(\sqrt{-5})$. For each prime ideal P of K, consider the corresponding Artin symbol in Gal $(K(i)/K)$. Prove that this correspondence induces an isomorphism between the class group of K and the Galois group of $K(i)$ over K.

Q 5.
Let $f \in O_K[X]$ be an irreducible, monic polynomial where K is a number field. Use the Frobenius density theorem for the Galois group of the splitting field of f over K to deduce that there are infinitely many prime ideals P in O_K such that f has no roots mod P.

OR

Let L/K be a Galois extension of number fields and let m be a modulus for K containing all the ramified places. Consider the group I_L^m which is the subgroup of the class group of L generated by the primes outside those lying over primes in m. Prove that as a module over Gal (L/K), the first cohomology group $H^1(I_L^m)$ is trivial.